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Wavelet representations and Fock space on positive matrices☆

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Abstract

We show that every biorthogonal wavelet determines a representation by operators on Hilbert space satisfying simple identities, which captures the established relationship between orthogonal wavelets and Cuntz-algebra representations in that special case. Each of these representations is shown to have tractable finite-dimensional co-invariant doubly cyclic subspaces. Further, motivated by these representations, we introduce a general Fock-space Hilbert space construction which yields creation operators containing the Cuntz–Toeplitz isometries as a special case.

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In this paper, we wish to establish a connection between biorthogonal wavelets on the one hand [16], and representation theory for operators on Hilbert space on the other [9,18]. This is accomplished by showing that each of these wavelets yields a collection of operators acting on Hilbert space which satisfy simple identities, and which contain the Cuntz relations [15] as a special case. In fact, this new relationship collapses to the now well-known connection between orthogonal wavelets and representations of the Cuntz C^*-algebra in that special case [10]. Our second goal is

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to develop a framework for studying this new class of representations. Toward this end, we introduce a general Fock space Hilbert space construction which reduces to unrestricted Fock space in the familiar cases. Indeed, the natural creation operators we get can be thought of as an analogue of the Cuntz–Toeplitz creation operators to this more general setting. We regard this construction and the creation operators determined by it as interesting objects of study in their own right. Finally, our hope is that this paper will lead to further study of the relationships and objects introduced here.

1. Introduction

In recent years we have seen several operator-theoretic approaches to wavelet theory, e.g., [3,10,21,23,29,35]. Typically, they involve representing wavelets of a particular type by operators on infinite-dimensional Hilbert space. They have had success because often the operators satisfy simple identities and hence lend themselves to investigation. From an operator theory and operator algebra point of view, these approaches often yield interesting new classes of examples to work with, and can open up new areas of study [7,8,18,19,22,23]. From the wavelet perspective, operator theory can be used to study the fundamental analysis/synthesis problem for wavelets, i.e., the transformation from functions in the Hilbert space $L^2(\mathbb{R})$ to wavelet coefficients in $\ell^2(\mathbb{Z})$. For instance, the paper [23] includes an application of work from [18] on *free semigroup operator algebras* to obtain a lucid characterization of when the data going into an *orthogonal wavelet* is minimal. Pioneering early papers which suggest use of operator theory and representations of groups and algebras in wavelet analysis include [3,4,6,14,17,21,28,31,35,38]. In this paper, we introduce an operator-theoretic approach for the study of *biorthogonal wavelets*. We will discuss the particulars of such wavelets and this approach in the next section.

We now discuss some of the basics of the wavelet cum operator approach, and we will continue this discussion in the next section. We use standard wavelet nomenclature from such texts as [7,16]. Wavelet theory is centered around the action of the integers $\mathbb{Z}$ on the Hilbert space $L^2(\mathbb{R})$ by the translations $f \mapsto f(x-k)$, $k \in \mathbb{Z}$, and by a fixed *scaling operator*

$$U: f \mapsto \frac{1}{\sqrt{N}} f\left(\frac{x}{N}\right) \quad \text{for } f \in L^2(\mathbb{R}). \tag{1}$$

The existence of a *resolution subspace* $\mathscr{V}$ in $L^2(\mathbb{R})$ which is invariant for translation by $\mathbb{Z}$ and also for the scaling operator U, is equivalent to the existence of a cyclic subspace under translation with cyclic vector $\varphi \in L^2(\mathbb{R})$ which in an important special case can be shown to satisfy

$$\frac{1}{\sqrt{N}} \varphi\left(\frac{x}{N}\right) = \sum_{k \in \mathbb{Z}} a_k \varphi(x-k), \tag{2}$$

for some scalars a_k. When such a φ exists, it is called a *scaling function* and generates in an algorithmic fashion a set of wavelet basis functions for $L^2(\mathbb{R})$. There is also an explicit correspondence between the scaling function and its so-called *filter functions* $\{m_i : 0 \leqslant i \leqslant N-1\}$, which in turn define the operators $\{S_i\}$ we have been studying. For the biorthogonal wavelets, there will be two functions φ, $\tilde{\varphi}$, as well as two sets of related filter functions $\{m_i, \tilde{m}_j\}$ and sets of operators $\{S_i, \tilde{S}_j\}$ which we are interested in. One of the attractive properties of this whole set up is that the various correspondences are explicit; there are formulae which allow us to go back and forth between each of the settings. We expand on these correspondences below, and in the next section.

Just as passing to a resolution subspace $L^2(\mathbb{R}) \to \mathscr{V}$ is a reduction of the initial analysis/synthesis problem, we will aim for a setup which is effective for computations, and so a Hilbert space isometry $\mathscr{V} \to \ell^2(\mathbb{Z})$ is desirable, and a further reduction to a much smaller subspace, which converts the original problem into one of manipulating sequences, i.e., vectors in $\ell^2(\mathbb{Z})$. But we have $\ell^2(\mathbb{Z}) \simeq L^2(\mathbb{T})$, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, by virtue of the Fourier transform, and it will be convenient to couch the operator theory in terms of the *function space* $L^2(\mathbb{T})$. Setting $N = 2$ for simplicity, it turns out that it is possible to find functions m_0, m_1 on $\mathbb{T}$, and functions φ, ψ in $L^2(\mathbb{R})$ with $\varphi \in \mathscr{V}$ and $U\psi \in \mathscr{V}$, such that the quadrature wavelet problem takes the following form: Let $S_i h(z) = m_i(z)h(z^2)$, $h \in L^2(\mathbb{T})$, $i = 0, 1$, and let $\hat{S}_i$ be the corresponding operators on $\ell^2(\mathbb{Z})$ with adjoints $\hat{S}_i^*$. Introduce for $c = \{c_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$,

$$c * f(x) = \sum_{k\in\mathbb{Z}} c_k f(x-k), \quad x \in \mathbb{R}. \tag{3}$$

Then under suitable conditions on m_0 and m_1, it is possible to get the following representation of the scaling operator (called the resolution/detail decomposition):

$$\mathscr{V} \ni c * \varphi = U[(\hat{S}_0^* c) * \varphi] + U[(\hat{S}_1^* c) * \psi],$$

where U is the scaling operator (1) for $N = 2$. From this we can then deduce an algorithmic approach to the analysis/synthesis problem of wavelets, i.e.,

$$L^2(\mathbb{R}) \ni f = \sum_{j,k\in\mathbb{Z}} c_{j,k}\psi_{j,k}, \quad f \leftrightarrow (c_{j,k}),$$

where $\psi_{j,k}(x) = 2^{\frac{j}{2}}\psi(2^j x - k)$ is a wavelet basis for $L^2(\mathbb{R})$.

Hence the wavelet problem has been translated into one for a different operator system, not in $L^2(\mathbb{R})$ but rather in the sequence space $\ell^2(\mathbb{Z})$. The operators $F_i = \hat{S}_i^*$ are known in signal processing as subband filters. When they are further assumed to satisfy identities (i) of Theorem 2.7, they are called quadrature mirror filters. Identities (i) are also called the Cuntz relations. Realizing that the Cuntz relations are in fact the quadrature subband-filter identities, therefore makes the connection to the so-called 'pyramid algorithm' of wavelet theory, see [16]; in other words, to the problem of discretizing signals using wavelets.

The next reduction is then to try to determine the latter problem from an equivalent one which involves only a finite-dimensional subspace $\mathscr{K}$ in $\ell^2(\mathbb{Z})$, or equivalently in $L^2(\mathbb{T})$. In other words, the goal is to discern the actions of the S_i-operators, hence the structure of the resolution and the wavelet itself, simply by examining their actions on a tractable finite-dimensional subspace. This has been accomplished for orthogonal wavelets when the scaling function φ has compact support. The space $\mathscr{K}$ is called the *anchor subspace* and it has been studied in separate and independent earlier papers by the coauthors [7,18,23,29]. We will show here that this can be done effectively for biorthogonal wavelets as well.

The next section begins with a discussion of the general method used to represent orthogonal wavelets as operators on Hilbert space satisfying the Cuntz relations. In particular, we recall the equivalence between (i) orthogonal wavelets of scale N, (ii) certain representations of the Cuntz C^*-algebra $\mathcal{O}_N$, and (iii) *matricial loops*, i.e., the group of all bounded measurable functions from the torus $\mathbb{T}$ into the unitary group $\mathrm{U}_N(\mathbb{C})$. We then prove a corresponding result for biorthogonal wavelets, including an analogous matrix perspective which involves *invertible loops*, i.e., the larger non-compact group of all bounded measurable functions from $\mathbb{T}$ into the general linear group ($= \mathrm{GL}_N(\mathbb{C})$). The link between a wavelet and its loop is provided by the filter functions.

As for the orthogonal wavelet representations [18,23,29], every biorthogonal wavelet representation is shown to have tractable finite-dimensional co-invariant cyclic subspaces. The finite dimensionality of the subspace requires that the wavelet be of compact support. This is the content of Section 3. In particular, the representation can be recovered, in a spatial sense, from these finite-dimensional anchor subspaces. We remark on how this can be regarded as a weak dilation theorem for the operators determined by the biorthogonal wavelet. We also find a striking relationship between the invertible-loop matrix entries and the operators $\{S_i, \tilde{S}_j\}$ which determine a biorthogonal representation. This gives us motivation for the Fock space approach.

In Sections 4 and 5, we introduce a general Fock space Hilbert space construction which is motivated by the biorthogonal wavelet representations. Every completely positive map from complex matrices to the bounded operators on a Hilbert space (equivalently, every positive matrix with operator entries) determines a 'twisted' Fock-space structure, which in turn yields natural *creation operators*. The Cuntz–Toeplitz isometries acting on unrestricted Fock space are recovered in two different ways; through the orthogonal class, and then from how the orthogonal class sits inside the biorthogonal class. We describe the actions of the creation operators which the spatial construction yields, and characterize when they are bounded operators in terms of the completely positive map. Our initial goal with this construction was to find analogues of the Cuntz–Toeplitz isometries for the biorthogonal wavelet representations. In any event, we regard these new creation operators as interesting objects of study in their own right, and our hope is that this paper will lead to further study of them.

2. Wavelet representations and operator identities

We shall consider a family of representations associated with wavelets, and relate them to representations of operator identities, which have the Cuntz relations as a special case. In the simplest case, the associated operators on $L^2(\mathbb{T})$ in the representations have the form

$$S\colon f \mapsto m(z) f(z^N), \quad f \in L^2(\mathbb{T}), \tag{4}$$

where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ is the usual torus, and $L^2(\mathbb{T})$ is the Hilbert space defined from the Haar measure μ on $\mathbb{T}$. The number $N \in \{2, 3, 4, \dots\}$ will be fixed, and the function $m \in L^\infty(\mathbb{T})$ determines the operator.

Using the isomorphism $\ell^2(\mathbb{Z}) \cong L^2(\mathbb{T})$ of Fourier series, we note that operator (4) may also be realized as acting on sequences $\mathbf{x}$ as follows:

$$(S\mathbf{x})_i = \sum_{j \in \mathbb{Z}} c_{i-Nj} x_j \quad \text{for } i \in \mathbb{Z} \text{ and } \mathbf{x} = (x_j)_{j \in \mathbb{Z}}, \tag{5}$$

where $\{c_j\}_{j \in \mathbb{Z}}$ forms the Fourier expansion of $m(z)$. The corresponding $\infty \times \infty$ matrix has the following form (the case $N = 2$):

$$\begin{array}{ccccccccc}
 & & & & \vdots & & & & \\
 & c_3 & c_1 & c_{-1} & c_{-3} & c_{-5} & c_{-7} & c_{-9} & \\
 & c_4 & c_2 & c_0 & c_{-2} & c_{-4} & c_{-6} & c_{-8} & \\
 & c_5 & c_3 & c_1 & c_{-1} & c_{-3} & c_{-5} & c_{-7} & \\
\cdots & c_6 & c_4 & c_2 & c_0 & c_{-2} & c_{-4} & c_{-6} & \cdots \\
 & c_7 & c_5 & c_3 & c_1 & c_{-1} & c_{-3} & c_{-5} & \\
 & c_8 & c_6 & c_4 & c_2 & c_0 & c_{-2} & c_{-4} & \\
 & c_9 & c_7 & c_5 & c_3 & c_1 & c_{-1} & c_{-3} & \\
 & c_{10} & c_8 & c_6 & c_4 & c_2 & c_0 & c_{-2} & \\
 & c_{11} & c_9 & c_7 & c_5 & c_3 & c_1 & c_{-1} & \\
 & & & & \vdots & & & &
\end{array}$$

It is down-slanted with slope 2, and it is called a *slanted Toeplitz matrix*. Its spectral properties are in some ways analogous to those of Toeplitz matrices, and in other ways completely different; see [7 and 12]. In this form, it is known as the *subdivision operator*, and it is used in numerical analysis, see e.g., [7,12,28,38]. The translation from (4) to (5) may be carried out by the usual Fourier series representation,

$$m(z) = \sum_{k \in \mathbb{Z}} c_k z^k, \quad c_k = \int_{\mathbb{T}} z^{-k} m(z)\, d\mu(z) \tag{6}$$

and

$$f(z) = \sum_{k \in \mathbb{Z}} x_k z^k, \quad z \in \mathbb{T}, \; x_k = \int_{\mathbb{T}} z^{-k} f(z)\, d\mu(z).$$

The numbers (c_k) are called the *masking coefficients* for the subdivision. The adjoint of (4) is known as the transfer operator, alias the Perron–Frobenius–Ruelle operator [7], and its action is readily described. The proof of the following lemma is routine.

Lemma 2.1. *If S is an operator on $L^2(\mathbb{T})$ with $(Sf)(z) = m(z)f(z^N)$, then S^* is given by*

$$(S^* f)(z) = \frac{1}{N} \sum_{w^N = z} \overline{m(w)} f(w).$$

Let us expand on the connection between the operators S on $L^2(\mathbb{T})$ and the underlying function theory on $L^2(\mathbb{R})$.

Proposition 2.2. *Let $\varphi \in L^2(\mathbb{R})$ satisfy* (2). *Define the operator W_φ: $L^2(\mathbb{T}) \to L^2(\mathbb{R})$ by*

$$(W_\varphi f)(x) = \sum_{k \in \mathbb{Z}} x_k \varphi(x-k), \quad x_k = \hat{f}(k) = \int_{\mathbb{T}} z^{-k} f(z)\, d\mu(z), \quad k \in \mathbb{Z}.$$

Then

$$\frac{1}{\sqrt{N}} (W_\varphi f)\left(\frac{x}{N}\right) = (W_\varphi S f)(x) \quad \textit{for } x \in \mathbb{R}, \tag{7}$$

where S is the operator defined by (4) *and* (6).

Proof. Let φ, W_φ and S be as stated. Then for $f(z) = \sum_{k \in \mathbb{Z}} x_k z^k$, we conclude

$$\begin{aligned}
\frac{1}{\sqrt{N}} (W_\varphi f)\left(\frac{x}{N}\right) &= \frac{1}{\sqrt{N}} \sum_{k \in \mathbb{Z}} x_k \varphi\left(\frac{x}{N} - k\right) \\
&= \sum_{k \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} x_k c_j \varphi(x - Nk - j) \\
&= \sum_{l \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} x_k c_{l-Nk} \right) \varphi(x - l) = (W_\varphi S f)(x),
\end{aligned}$$

which follows from identity (5). This is the desired identity claimed in the proposition. Introducing the scaling operator of (1), identity (7) takes the equivalent form $UW_\varphi = W_\varphi S$, and we say that W_φ intertwines the two operators U and S. It is called the wave operator, and it plays a central role in computational harmonic analysis. $\square$

Remark 2.3. The conclusion of the proposition, and the usefulness of the S_i-operators, are relevant even if the scaling identity (2) does not have a solution in $L^2(\mathbb{R})$. This is demonstrated for example in [22,24] for the case $N=4$ with the following scaling identity

$$\varphi(x) = 2\varphi(4x) + 2\varphi(4x-2). \tag{8}$$

Rewriting (8) as

$$\int h(x)\,d\varphi(x) = \frac{1}{2}\left(\int h\left(\frac{x}{4}\right)\,d\varphi(x) + \int h\left(\frac{x+2}{4}\right)d\varphi(x)\right), \tag{9}$$

with h continuous, we see that (8) has a unique probability measure $d\varphi$ as its solution. It has Hausdorff dimension $HD[d\varphi] = \frac{1}{2}$, so it is a singular measure. Moreover, (8) does not have a solution φ in $L^2(\mathbb{R})\backslash\{0\}$.

As discussed in the previous section, the scaling function $\varphi \in L^2(\mathbb{R})$ for an orthogonal wavelet, if it exists, may be obtained as a solution to the equation

$$\varphi(x) = \sqrt{N}\sum_{k\in\mathbb{Z}} c_k\varphi(Nx-k), \tag{10}$$

but existence of $L^2(\mathbb{R})$ solutions requires special assumptions on m, or equivalently (c_k), from (6), to which we now turn. In particular, we now obtain the equivalences which lead to wavelet representations.

Lemma 2.4. *We have the following three equivalences*:

(i) $m \in L^\infty(\mathbb{T})$ *if and only if* S *is bounded.*

(ii) $\frac{1}{N}\sum_{\substack{w\in\mathbb{T}\\ w^N=z}} |m(w)|^2 = 1$ a.a. $z\in\mathbb{T}$ *if and only if* S *is isometric.*

(iii) *If* $m_1, m_2 \in L^\infty$ *are given, then the corresponding operators* S_1 *and* S_2 *have orthogonal ranges if and only if for* a.a. $z\in\mathbb{T}$,

$$\frac{1}{N}\sum_{w^N=z} \overline{m_1(w)}\,m_2(w) = 0.$$

Remark 2.5. S^*S is a multiplication operator by $\frac{1}{N}\sum_{\substack{w\in\mathbb{T}\\ w^N=z}} |m(w)|^2$ but SS^* is not; in fact S is not normal, nor even hyponormal.

Proof of Lemma 2.4. The equivalence in (i) is clear. For (ii), observe that for $f \in L^2(\mathbb{T})$,

$$\begin{aligned}(S^*Sf)(z) &= \frac{1}{N}\sum_{w^N=z} \overline{m(w)}(Sf)(w)\\ &= \frac{1}{N}\sum_{w^N=z} \overline{m(w)}\, m(w) f(w^N)\\ &= \frac{1}{N}\sum_{w^N=z} \overline{m(w)}\, m(w) f(z)\\ &= \frac{1}{N}\sum_{w^N=z} |m(w)|^2 f(z).\end{aligned}$$

In particular, $S^*S = I$ precisely when the condition on $m(z)$ in (ii) is satisfied. Finally, if S_1 and S_2 are given by m_1 and m_2, a similar calculation shows that

$$(S_1^* S_2 f)(z) = \frac{1}{N}\sum_{w^N=z} \overline{m_1(w)}\, m_2(w) f(z),$$

whence equivalence (iii) becomes apparent.

Before continuing, let us set aside the basic definitions from wavelet theory which we need [16].

Definition 2.6. By a *wavelet of scale N* we mean a finite set of functions ψ_i, $i = 1, \dots, N-1$, in $L^2(\mathbb{R})$ such that the family

$$\psi_{i,j,k}(x) := N^{\frac{j}{2}} \psi_i(N^j x - k), \quad j, k \in \mathbb{Z},$$

satisfies

$$\langle f | f \rangle = \int_{\mathbb{R}} |f(x)|^2 \, dx = \sum_{i,j,k} |\langle f | \psi_{i,j,k} \rangle_{L^2(\mathbb{R})}|^2$$

for all $f \in L^2(\mathbb{R})$.

It is an *orthogonal wavelet* when the family $\psi_{i,j,k}$ forms an orthonormal basis for $L^2(\mathbb{R})$, equivalently, when every $||\psi_{i,j,k}|| = 1$. For such wavelets there is a 1–1 and explicit correspondence between the family $(\psi_i)_{i=1}^{N-1}$ together with the associated scaling function φ, and systems of so-called *wavelet filter functions* $(m_i)_{i=0}^{N-1}$ which are characterized by condition (ii) of Theorem 2.7 (see [9,10,23]).

More generally, a *biorthogonal wavelet* consists of two families $\{\psi_i\}$ and $\{\tilde{\psi}_i\}$ of $N-1$ functions in $L^2(\mathbb{R})$ such that

$$\langle f | g \rangle = \sum_{i,j} \overline{\langle \psi_{i,j,k} | f \rangle} \langle \tilde{\psi}_{i,j,k} | g \rangle \quad \text{for } f, g \in L^2(\mathbb{R}).$$

These wavelets also have associated filter functions $\{m_i\}$ and $\{\tilde{m}_i\}$ which satisfy the condition specified in (ii) of Theorem 2.8.

As a first consequence of the previous lemma we obtain the well-known method (for instance see [10,23]) of generating Cuntz-algebra representations from orthogonal wavelets. The *Cuntz algebra* $\mathcal{O}_N$ is the universal C^*-algebra generated by the relations in (i) of Theorem 2.7. It has been studied extensively by operator algebraists since the work [15].

Theorem 2.7. *The following three conditions are equivalent when the functions* $m_0, \dots, m_{N-1} \in L^\infty(\mathbb{T})$ *are given and operators* $S_0, \dots, S_{N-1}$ *are defined by* $S_i f(z) = m_i(z) f(z^N)$.

(i)
$$\begin{cases} S_i^* S_j = \delta_{i,j} I, \\ \sum_{i=0}^{N-1} S_i S_i^* = I. \end{cases}$$

(ii) *The functions* $m_0, \dots, m_{N-1}$ *on the torus* $\mathbb{T}$ *are the filter functions for an orthogonal wavelet of scale* N. *In other words, the matrix*
$$\frac{1}{\sqrt{N}} (m_k(e^{i\frac{2\pi l}{N}} z))_{k,l=0}^{N-1}$$
is in $\mathrm{U}_N(\mathbb{C})$ a.a. $z \in \mathbb{T}$.

(iii) $A_{k,l}(z) = \frac{1}{N} \sum_{w^N = z} w^{-l} m_k(w)$ *are the entries of a loop* $\mathbb{T} \to \mathrm{U}_N(\mathbb{C})$, *i.e., a matrix function*
$$A(z) = (A_{k,l}(z)) \in \mathrm{U}_N(\mathbb{C}) \quad \text{a.a. } z \in \mathbb{T}.$$

Proof. The two Cuntz identities in (i) correspond to the orthonormality of the rows and columns in the matrices of (ii). Indeed, the previous lemma shows that the S_i being isometries with pairwise orthogonal ranges is the same as the rows being orthonormal. On the other hand, for $f \in L^2(\mathbb{T})$ we have

$$\begin{aligned} \left\langle \sum_{i=0}^{N-1} S_i S_i^* f \,\middle|\, f \right\rangle &= \sum_{i=0}^{N-1} \langle S_i^* f \mid S_i^* f \rangle \\ &= \sum_{i=0}^{N-1} \frac{1}{N^2} \int_{\mathbb{T}} \sum_{w^N = z = w'^N} \overline{m_i(w)}\, m_i(w') f(w) \overline{f(w')}\, d\mu(z) \\ &= \frac{1}{N} \int_{\mathbb{T}} \sum_{w^N = z = w'^N} \underbrace{\left(\frac{1}{N} \sum_{i=0}^{N-1} \overline{m_i(w)}\, m_i(w') \right)}_{= \delta_{w,w'}} f(w) \overline{f(w')}\, d\mu(z). \end{aligned}$$

However, we can write $\|f\|_2^2 = \frac{1}{N}\int_{\mathbb{T}} \sum_{w^N=z} |f(w)|^2\, d\mu(z)$. Hence the identity $\sum_{i=0}^{N-1} S_i S_i^* = I$ is equivalent to the condition

$$\frac{1}{N}\sum_{i=0}^{N-1} \overline{m_i(w)}\, m_i(w') = \delta_{w,w'},$$

for Nth roots w and w' of a.a. $z \in \mathbb{T}$. This is equivalent to column orthonormality. For the equivalence of (ii) and (iii) consider the following calculation:

$$\begin{aligned}\sum_{k=0}^{N-1} A_{i,k}(z)\,\overline{A_{j,k}(z)} &= \frac{1}{N^2}\sum_k \sum_{w^N=z=w'^N} w^{-k} m_i(w) w'^k \overline{m_j(w')}\\ &= \frac{1}{N}\sum_{w,w'}\left(\frac{1}{N}\sum_{k=0}^{N-1} (w^{-1}w')^k\right) m_i(w)\,\overline{m_j(w')}\\ &= \frac{1}{N}\sum_{w^N=z} m_i(w)\,\overline{m_j(w)}.\end{aligned}$$

Thus the matrix $A(z)$ is unitary precisely when the matrix in (ii) is unitary. □

There are a number of advantages obtained by using the matrix approach given by the $A(z)$, including the fact that the filter functions m_i can be recovered from A (see Section 3). For our purposes, this approach is helpful when considering co-invariant subspaces, and it provides motivation for our Fock-space construction. Now let us turn to the new result here: namely, every biorthogonal wavelet yields operators on Hilbert space satisfying simple identities which contain the Cuntz relations in the special case of orthogonal wavelets. There is an analogous matrix approach as well.

Theorem 2.8. *The following conditions are equivalent when the functions* $m_0, \dots, m_{N-1}$, $\tilde{m}_0, \dots, \tilde{m}_{N-1}$, *and the corresponding operators* $S_i f(z) := m_i(z) f(z^N)$, $\tilde{S}_i f(z) := \tilde{m}_i(z) f(z^N)$ *are given*:

(i)
$$\begin{cases} S_i^* \tilde{S}_j = \delta_{i,j} I, \\ \sum_{i=0}^{N-1} S_i \tilde{S}_i^* = I. \end{cases}$$

(ii) *The functions* $m_0, \dots, m_{N-1}$, $\tilde{m}_0, \dots, \tilde{m}_{N-1}$ *are the filter functions for a biorthogonal wavelet of scale* N. *In other words, the matrices*

$$\frac{1}{\sqrt{N}}\left(m_k\left(e^{i\frac{2\pi l}{N}} z\right)\right)_{k,l=0}^{N-1} \quad \textit{and} \quad \frac{1}{\sqrt{N}}\left(\tilde{m}_k\left(e^{i\frac{2\pi l}{N}} z\right)\right)_{k,l=0}^{N-1}$$

belong to $\mathrm{GL}_N(\mathbb{C})$ *and the adjoint of one is the inverse of the other for* a.a. $z \in \mathbb{T}$.

(iii) *The two matrix functions A and $\tilde{A}$ with entries*

$$A_{k,l}(z) = \frac{1}{N} \sum_{w^N=z} w^{-l} m_k(w),$$

$$\tilde{A}_{k,l}(z) = \frac{1}{N} \sum_{w^N=z} w^{-l} \tilde{m}_k(w)$$

satisfy

$$\sum_{k=0}^{N-1} \overline{A_{k,i}(z)}\, \tilde{A}_{k,j}(z) = \delta_{i,j}, \quad \text{a.a. } z \in \mathbb{T},$$

i.e.,

$$A^* \tilde{A} = I \quad \textit{pointwise}, \ \text{a.a. } z \in \mathbb{T},$$

or

$$\tilde{A} = A^{*-1},$$

where the function $z \to A^*(z)$ denotes the adjoint matrix function mapping $\mathbb{T} \to \mathrm{GL}_N(\mathbb{C})$.

Proof. With small adjustments we can follow the lines of the previous proof. From the lemma, it follows that the condition $S_i^* \tilde{S}_j = \delta_{i,j} I$ is equivalent to the identity

$$\frac{1}{N} \sum_{w^N=z} \overline{m_i(w)} \tilde{m}_j(w) = \delta_{i,j},$$

for $0 \leqslant i,j \leqslant N-1$ and a.a. $z \in \mathbb{T}$, while the identity $\sum_{i=0}^{N-1} S_i \tilde{S}_i^* = I$ is a restatement of

$$\frac{1}{N} \sum_{i=0}^{N-1} \overline{m_i(w)} \tilde{m}_i(w') = \delta_{w,w'},$$

for Nth roots w and w' of a.a. $z \in \mathbb{T}$. Thus the first two conditions are equivalent. Finally, we can see conditions (ii) and (iii) are equivalent by following the computation in the previous proof with $\tilde{A}_{i,k}(z)$ replacing $A_{i,k}(z)$. $\square$

Example 2.9. The matrix functions A: $\mathbb{T} \to \mathrm{GL}_N(\mathbb{C})$ of Theorems 2.7 and 2.8 might be constant even though the filter functions $\{m_i\}_{i=0}^{N-1}$ are non-constant. If $N = 2$ and

$$\varphi(x) = \begin{cases} 1, & 0 \leqslant x < 1, \\ 0, & \text{other } x \in \mathbb{R} \end{cases}$$

is the scaling function for the Haar wavelet, then $m_0(z) = \frac{1}{\sqrt{2}}(1+z)$ and $A(z) = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right] \in \mathrm{U}_2(\mathbb{C})$.

For the example in Remark 2.3, $N = 4$, $m_0(z) = 1 + z^2$ and

$$A(z) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

is an admissible non-unitary matrix function, here mapping $\mathbb{T} \to \mathrm{GL}_4(\mathbb{C})$. This yields a biorthogonal system, where the biorthogonality is encoded in the duality between measures and continuous functions on $[0,1] \subset \mathbb{R}$. To get scaling functions belonging to $L^2(\mathbb{R})$, m_0 must satisfy $\frac{1}{N}\sum_{w^N=z} |m_0(w)|^2 \leqslant 1$ [7]; and one checks that $m_0(z) = 1 + z^2$ violates this. In fact, $\frac{1}{4}\sum_{w^4=z} |m_0(w)|^2 = 2$.

Our convention for the filter function $m_0(z) = \sum_k c_k z^k$ is that the coefficients (c_k) are the masking numbers in (10). Hence we assume that $m_0(1) = \sqrt{N}$ where N is the scaling number from (10). Introducing the 2π-periodic variant of m_0, i.e.,

$$m_0(t) := m_0(e^{-it}),$$

we get the product formula for the Fourier transform

$$\hat{\varphi}(t) = \int_{\mathbb{R}} e^{-itx}\, d\varphi(x)$$

from (10) in the form

$$\hat{\varphi}(t) = \prod_{j=1}^{\infty} \frac{m_0(t/N^j)}{\sqrt{N}} \quad \text{for } t \in \mathbb{R}. \tag{11}$$

This works even if $d\varphi$ is only a tempered distribution. If

$$\frac{1}{N}\sum_{k=0}^{N-1}\left| m_0\left(t + \frac{k2\pi}{N}\right)\right|^2 \leqslant 1,$$

it follows that the infinite product in (11) is in $L^2(\mathbb{R})$.

Definition 2.10. We refer to the relations in (i) of Theorem 2.8 as the *biorthogonal relations*. Further, any system $\{S_i\}$ satisfying the relations in (i) of Theorem 2.7 clearly determines a representation of $\mathcal{O}_N$, what we call an *orthogonal wavelet representation*. Similarly, we will refer to a system $\{S_i, \tilde{S}_j\}$ satisfying the biorthogonal relations as a *biorthogonal wavelet representation*.

Remark 2.11. There are a number of properties which can be derived from the biorthogonal relations. For instance, a simple matrix argument shows there are no finite-dimensional representations of them. Further, there are some nice reductions which can be made on words in the generators $\{S_i, S_i^*, \tilde{S}_i, \tilde{S}_i^*\}$. However, overall there is little we can say about operators which satisfy these relations in full generality. Fortunately, the biorthogonal wavelet representations of the biorthogonal relations have some additional properties. As we shall see in the next section they have tractable finite-dimensional co-invariant cyclic subspaces, and the generators satisfy other helpful relations.

The following simple lemma (see, e.g., [9]) will help clarify the discussion below:

Lemma 2.12. *Let $R := R_N$ be the average operator*

$$Rf(z) := \frac{1}{N} \sum_{\substack{w \in \mathbb{T} \\ w^N = z}} f(w).$$

It is contractive in $L^\infty(\mathbb{T})$, and co-isometric in $L^2(\mathbb{T})$. Setting $S_0 f(z) = f(z^N)$, we have $R = S_0^*$, where the $*$ refers to the adjoint operation relative to $L^2(\mathbb{T})$, and

$$\ker(R)^\perp = S_0 L^2(\mathbb{T}).$$

Proof. Let $e_k(z) := z^k$, $z \in \mathbb{T}$, $k \in \mathbb{Z}$. This is the standard Fourier basis in $L^2(\mathbb{T})$. Using duality for the finite cyclic group $\mathbb{Z}/N\mathbb{Z} \cong \{0, 1, \dots, N-1\}$, we then get

$$Re_k = \begin{cases} e_{k/N} & \text{if } k \equiv 0 \bmod N, \\ 0 & \text{if } k \not\equiv 0 \bmod N. \end{cases}$$

The remaining details are left to the reader; or see [9]. $\square$

Recall that if $L^2(\mathbb{T})$ is a module over $\mathscr{A}$, then functions $\{m_i\} \subseteq L^2(\mathbb{T})$ form a *module basis* for this module if given $f \in L^2(\mathbb{T})$ there is a unique expansion $f = \sum_i m_i a_i$ with $a_i \in \mathscr{A}$.

Corollary 2.13. *$\mathscr{A}_1 := S_0(L^\infty(\mathbb{T}))$ is a subalgebra of $L^\infty(\mathbb{T})$, and $L^2(\mathbb{T})$ is a module over $\mathscr{A}_1$ of module dimension N. In fact, the functions $m_0, \dots, m_{N-1}$ form a module basis for $L^2(\mathbb{T})$ over $\mathscr{A}_1$ if and only if they satisfy condition* (ii) *in Theorem* 2.7, *i.e., if and only if*

$$R(\bar{m}_i m_j) = \delta_{i,j} 1.$$

Proof. Since

$$R(\bar{m}_i m_j)(z) = \frac{1}{N} \sum_{w^N = z} \bar{m}_i(w) m_j(w)$$
$$= \frac{1}{N} \sum_{k=0}^{N-1} \bar{m}_i\left(e^{i\frac{2\pi k}{N}} z^{\frac{1}{N}}\right) m_j\left(e^{i\frac{2\pi k}{N}} z^{\frac{1}{N}}\right),$$

where $z^{\frac{1}{N}}$ is the principal branch of the Nth root, the orthogonality relations are clear. By (i)$\Leftrightarrow$(ii) in Theorem 2.7, we have

$$L^2(\mathbb{T}) \ni f(z) = \sum_{i=0}^{N-1} S_i S_i^* f(z) = \sum_{i=0}^{N-1} m_i(z)(S_i^* f)(z^N) \in \sum_{i=0}^{N-1} m_i \mathscr{A}_1$$

which shows that the orthogonality relations make $m_0, \ldots, m_{N-1}$ a module basis. $\square$

Corollary 2.14. *$\mathscr{A}_k := S_0^k(L^\infty(\mathbb{T}))$ is a subalgebra of $\mathscr{A}_{k-1}$ of module dimension N, and the module dimension of $L^2(\mathbb{T})$ over $\mathscr{A}_k$ is N^k. The corresponding module basis is*

$$b_{i_1, i_2, \ldots, i_k} := m_{i_1}(z) m_{i_2}(z^N) \cdots m_{i_k}\left(z^{N^{k-1}}\right).$$

Proof. Every $f \in L^\infty(\mathbb{T})$ satisfies

$$f = \sum_{i_1, \ldots, i_k} S_{i_1} \cdots S_{i_k} S_{i_k}^* \cdots S_{i_1}^* f.$$

Setting $f_{i_1, \ldots, i_k} := S_{i_k}^* \cdots S_{i_1}^* f$, we get

$$f(z) = \sum_{i_1, \ldots, i_k} b_{i_1, \ldots, i_k}(z) f_{i_1, \ldots, i_k}(z^{N^k}) \in \sum_{i_1, \ldots, i_k} b_{i_1, \ldots, i_k} \mathscr{A}_k. \qquad \square$$

3. Co-invariant subspaces

From the theory of wavelets [16], a compactly supported biorthogonal wavelet of scale N is determined by scaling functions φ and $\tilde{\varphi}$ which generate the associated $2N-2$ wavelet functions. Further, the functions $\varphi, \tilde{\varphi}$ are supported on the interval $[-Ng+1, Ng-1]$, where g is the *genus* of the wavelet. (We point out that the recent book [7] examines spaces of all such scaling functions.) In this case, the corresponding filter functions m_i and $\tilde{m}_i$ are Fourier polynomials of degree $Ng-1$, i.e., of the form $\sum_{k=-Ng+1}^{Ng-1} a_k z^k$. The numbers a_k are the wavelet masking coefficients, i.e., $\varphi(x) = \sum_k a_k \varphi(Nx-k)$. If $a_k = 0$ unless $0 \leqslant k \leqslant Ng-1$, then φ is supported in $[0, Ng-1]$. One of the advantages of using the matrix perspective given by the invertible loops is that this degree is considerably reduced for the functions

$A_{i,j}$, $\tilde{A}_{i,j}$. We begin by observing this fact, together with the method of recovering the filter functions (hence the $S, \tilde{S}$ system) from the invertible loops.

Lemma 3.1. *For $0 \leqslant i \leqslant N-1$, the filters $m_i(z)$, $\tilde{m}_i(z)$ are obtained from A, $\tilde{A}$ by*

$$m_i(z) = \sum_{j=0}^{N-1} A_{i,j}(z^N) z^j \quad \textit{and} \quad \tilde{m}_i(z) = \sum_{j=0}^{N-1} \tilde{A}_{i,j}(z^N) z^j.$$

Further, if each m_i, $\tilde{m}_i$ is a polynomial of degree at most $Ng-1$, then the $A_{i,j}$, $\tilde{A}_{i,j}$ have degree at most $g-1$.

Proof. The first claim simply follows from the computation

$$\begin{aligned}\sum_j A_{i,j}(z^N) z^j &= \sum_j \left(\frac{1}{N} \sum_{w^N = z^N} m_i(w) w^{-j} \right) z^j \\ &= \sum_{w^N = z^N} m_i(w) \delta_{w,z} = m_i(z).\end{aligned}$$

The same computation works for $\tilde{m}_i$ and $\tilde{A}_{i,j}$. To verify the second claim, suppose $m_i(z) = \sum_{k=0}^{Ng-1} a_k^{(i)} z^k$. Then we have

$$\begin{aligned}A_{i,j}(z) &= \frac{1}{N} \sum_{w^N = z} m_i(w) w^{-j} \\ &= \sum_{|k| \leqslant Ng-1} a_k^{(i)} \left(\frac{1}{N} \sum_{w^N = z} w^{k-j} \right) \\ &= \sum_{|k| \leqslant Ng-1} a_k^{(i)} z^{\frac{k-j}{N}} \delta_{k-j \ (\mathrm{mod}\ N),0}.\end{aligned}$$

However, the quantity $\frac{k-j}{N}$ is bounded above by $g-1$, as required. $\square$

The biorthogonal wavelet representations are rather specialized in that they have tractable finite-dimensional, co-invariant subspaces which are also *doubly cyclic*. We shall discuss the significance of this fact in Remark 3.4. This comes as a direct consequence of the following result. The key technical device is that, as for the orthogonal wavelet representations [10], the actions of the adjoint operators on Fourier basis vectors can be computed directly.

Lemma 3.2. *Let $S = (S_0, \ldots, S_{N-1})$, $\tilde{S} = (\tilde{S}_0, \ldots, \tilde{S}_{N-1})$ be a compactly supported biorthogonal wavelet representation of genus g. Let $\{e_n : n \in \mathbb{Z}\}$ be the basis for $L^2(\mathbb{T})$*

given by $e_n(z) = z^n$ *and let*

$$\mathscr{K} = \operatorname{span}\{e_0, e_{-1}, \dots, e_{-Ng+1}\} \bigwedge \underset{i,j}{\operatorname{span}}\{\overline{A_{i,j}(z)}\, z^r;\ r \leqslant 0\}.$$

Then we have

$$S_i^* \mathscr{K} \subseteq \mathscr{K} \quad \textit{and} \quad \tilde{S}_i^* \mathscr{K} \subseteq \mathscr{K} \quad \textit{for } 0 \leqslant i \leqslant N-1. \tag{12}$$

(*Properties* (12) *are called* co-invariance.) *Further*, *for all* $n \in \mathbb{Z}$ *there is a* $K \geqslant 1$ *such that*

$$S_{i_1}^* \cdots S_{i_k}^* e_n \in \mathscr{K} \quad \textit{and} \quad \tilde{S}_{i_1}^* \cdots \tilde{S}_{i_k}^* e_n \in \mathscr{K}$$

for all $k \geqslant K$ *and all indices* $0 \leqslant i_1, \dots, i_k \leqslant N-1$.

Proof. We first compute the action of S_i^* on a typical basis vector e_n. From the previous lemma we have

$$\begin{aligned} S_i^* e_n(z) &= \frac{1}{N} \sum_{w^N = z} \overline{m_i(w)} w^n \\ &= \sum_{j=0}^{N-1} \overline{A_{i,j}(z)} \left(\frac{1}{N} \sum_{w^N = z} w^{n-j} \right) \\ &= \sum_{j=0}^{N-1} \overline{A_{i,j}(z)}\, z^{\frac{n-j}{N}} \delta_{n-j \ (\mathrm{mod}\ N),0}. \end{aligned}$$

Of course, only one term in this sum is non-zero. Recall that the $A_{i,j}(z)$ are of degree at most $g-1$. Thus, if $0 \leqslant n \leqslant Ng-1$, it follows that $S_i^* e_{-n}$ is the complex conjugate of a polynomial of degree at most $Ng-1$. This says precisely that $S_i^* e_{-n} \in \mathscr{K}$.

It remains to check that the adjoints 'pull back' basis vectors into $\mathscr{K}$. The pattern becomes clear after two applications of adjoints. Let $A_{p,q}(z) = \sum_{l=0}^{g-1} A_l^{(p,q)} z^l$ for $0 \leqslant p, q \leqslant N-1$. Let $n \in \mathbb{Z}$ and set $j_0 \equiv n \ (\mathrm{mod}\ N)$ with $0 \leqslant j_0 \leqslant N-1$. Then we have

$$\begin{aligned} \overline{S_p^* S_i^* e_n}(z) &= \frac{1}{N} \sum_{w^N = z} m_p(w) A_{i,j_0}(w) w^{-\frac{n-j_0}{N}} \\ &= \frac{1}{N} \sum_{w} \left(\sum_{q} A_{p,q}(z) w^q \right) \left(\sum_{l} A_l^{(i,j_0)} w^l \right) w^{-\frac{n-j_0}{N}} \\ &= \sum_{q,l} A_{p,q}(z) A_l^{(i,j_0)} \left(\frac{1}{N} \sum_{w^N = z} w^{q+l-\frac{n-j_0}{N}} \right) \\ &= \sum_{q,l} A_l^{(i,j_0)} \delta_{q+l-\frac{n-j_0}{N} \ (\mathrm{mod}\ N),0} A_{p,q}(z) z^{\frac{q+l-\frac{n-j_0}{N}}{N}}. \end{aligned}$$

From this computation we see that vectors $S_{i_1}^* \cdots S_{i_k}^* e_n$ belong to the span of vectors of the form $\overline{A_{p,q}(z)} z^r$, where the absolute value of the powers r decreases steadily as k increases. It follows that $S_{i_1}^* \cdots S_{i_k}^* e_n$ will belong to $\mathscr{K}$ when k is large enough, and co-invariance means it will stay there.

We have carried out the analysis on the S_i^*, but the same proof works for the $\tilde{S}_i^*$. From Lemma 2.1 they have analogous formulae, and from the discussion at the start of this section the same compact support yields the same summation limits throughout. $\square$

Theorem 3.3. *Let $S = (S_0, \ldots, S_{N-1})$, $\tilde{S} = (\tilde{S}_0, \ldots, \tilde{S}_{N-1})$ be a compactly supported biorthogonal wavelet representation on $\mathscr{H} = L^2(\mathbb{T})$. Then there is a finite-dimensional subspace $\mathscr{K}$ which is co-invariant and doubly-cyclic for the representation. In other words,*

(i) $S_i^* \mathscr{K} \subseteq \mathscr{K}$ *and* $\tilde{S}_i^* \mathscr{K} \subseteq \mathscr{K}$ *for* $0 \leqslant i \leqslant N-1$,
(ii) $\bigvee_{\substack{i_1, \ldots, i_k \\ k \geqslant 1}} S_{i_1} \cdots S_{i_k} \mathscr{K} = \mathscr{H} = \bigvee_{\substack{i_1, \ldots, i_k \\ k \geqslant 1}} \tilde{S}_{i_1} \cdots \tilde{S}_{i_k} \mathscr{K}$.

Proof. The subspace $\mathscr{K}$ from Lemma 3.2 provides the candidate. The only thing left to show is cyclicity. But from the biorthogonal relations, for $k \geqslant 1$ we have

$$\sum_{0 \leqslant i_1, \ldots, i_k \leqslant N-1} S_{i_1} \cdots S_{i_k} \tilde{S}_{i_k}^* \cdots \tilde{S}_{i_1}^* = I. \tag{13}$$

In particular, when this identity is applied to Fourier basis vectors, the previous lemma yields (ii) for S. Specifically, let $n \in \mathbb{Z}$. Using then Lemma 3.2, we may pick some $k \in \mathbb{N}$, depending on n, such that

$$\tilde{S}_{i_k}^* \cdots \tilde{S}_{i_1}^* e_n \in \mathscr{K} \quad \text{for all } i_1, \ldots, i_k.$$

An application of (13) then yields $e_n \in \bigvee_{i_1, \ldots, i_k} S_{i_1} \cdots S_{i_k} \mathscr{K}$. The result follows since the closed span of the monomials e_n, $n \in \mathbb{Z}$, is $L^2(\mathbb{T})$. Finally, taking the adjoint of this identity and applying the lemma again for $\tilde{S}$ completes the proof. $\square$

Remark 3.4. There is an entire structure theory for representations of $\mathcal{O}_N$ which are found to have finite-dimensional, co-invariant cyclic subspaces. Indeed, the recent paper [18] sets out a theory for decomposing such representations into tractable classes of irreducible subrepresentations. This paper was presented in the context of dilation theory, but it was observed in [23,29] that the orthogonal wavelet representations of $\mathcal{O}_N$ form a subclass of these representations.

For such a representation, let A_i be the compressions of the isometries S_i to a given finite-dimensional, co-invariant cyclic subspace. The crucial point in the analysis is that the finite-dimensional *minimal* A_i^*-invariant subspaces generate the irreducible subspaces for the representation. This came from an investigation into the completely positive map $\Phi(X) = \sum_{i=0}^{N-1} A_i X A_i^*$ determined by the A_i. Thus decomposing these representations, acting on infinite-dimensional space, amounts to computing for these finite-dimensional 'anchor' subspaces. In fact, the paper [29] shows these subspaces can be obtained through a relatively simple analysis of the map Φ, without *any* explicit reference to the compressions A_i.

There are obvious analogues of this theory for the biorthogonal setting, but one immediately confronts serious issues. The analogue of Φ would be a *completely bounded* unital map $\Phi(X) = \sum_{i=0}^{N-1} A_i X \tilde{A}_i^*$. But in the orthogonal-$\mathcal{O}_N$-completely positive setting, the key technical device is the unique dilation theory which abounds: namely, what is known as the Frahzo–Bunce–Popescu unique minimal isometric dilation of a row contraction [11,20,37], which is really a special case of Stinespring's unique dilation of a completely positive map to a C^*-homomorphism (see [36]). This allows us to go back and forth interchangeably between the A_i and S_i, as well as, respectively, the completely positive map and the endomorphism determined by these operators. In our more general setting dilations typically exist, but they are *not unique*. For instance, recall from Paulsen's 'off-diagonal technique' [36] how completely bounded maps are dilated: Every completely bounded map can be regarded as the off-diagonal corner of a completely positive map. Stinespring's dilation theorem gives a unique dilation of this map, which in turn yields a completely bounded homomorphism which dilates the completely bounded map. The problem is that the way in which the map is regarded as an off-diagonal corner is not unique (in fact an application of Arveson's matricial Hahn–Banach Theorem is involved [1,2]). There is also the issue of irreducibility here. It is not even clear what it should mean for an $S, \tilde{S}$ system to be irreducible.

Nonetheless, for the wavelet representations of the biorthogonal relations at least, Theorem 3.3 shows that a weaker spatial version of these dilation results is valid here. In particular, the representations can be recovered spatially from the compressions to particular finite-dimensional anchor subspaces. The reader may notice that the computations above can be strengthened to reduce the size of $\mathcal{K}$. In fact, it appears that the analogue here of Section 8 from [23] could be developed to obtain 'minimal' subspaces $\mathcal{L}$ and $\tilde{\mathcal{L}}$ of $\mathcal{K}$ defined, respectively, by A and $\tilde{A}$, which are co-invariant for S (respectively $\tilde{S}$) and cyclic for $\tilde{S}$ (respectively S). From [18,23], an orthogonal wavelet representation is irreducible exactly when there is a unique such $\mathcal{L}$. It would be interesting to know if there is an analogue of this fact for the $\mathcal{L}$ and $\tilde{\mathcal{L}}$ here.

We finish this section by discovering a striking relationship between the operators $\{S_i, \tilde{S}_j\}$ on the one hand, and the matrices A and $\tilde{A}$ on the other.

Lemma 3.5. *Let* $S = (S_0, \dots, S_{N-1})$, $\tilde{S} = (\tilde{S}_0, \dots, \tilde{S}_{N-1})$ *be a biorthogonal wavelet representation with invertible-loop matrix functions* A, $\tilde{A}$. *Then for* $f \in L^2(\mathbb{T})$ *and* $0 \leqslant i, j \leqslant N-1$ *we have*

(i) $S_i^* S_j f(z) = (AA^*)_{j,i}(z) f(z)$ *and*
(ii) $\tilde{S}_i^* \tilde{S}_j f(z) = (AA^*)_{j,i}^{-1}(z) f(z)$.

Proof. For the S_i we have the following computation:

$$\begin{aligned}
S_i^* S_j f(z) &= \frac{1}{N} \sum_{w^N = z} \bar{m}_i(w) m_j(w) f(z) \\
&= \frac{1}{N} \sum_{w^N = z} \sum_{k,l} \bar{A}_{i,k}(z) \bar{w}^k A_{j,l}(z) w^l f(z) \\
&= \sum_{k,l} \delta_{k,l} \bar{A}_{i,k}(z) A_{j,l}(z) f(z) \\
&= \sum_{k} \bar{A}_{i,k}(z) A_{j,k}(z) f(z) \\
&= (AA^*)_{j,i}(z) f(z).
\end{aligned}$$

A similar calculation shows that

$$\tilde{S}_i^* \tilde{S}_j f(z) = (\tilde{A}\tilde{A}^*)_{j,i}(z) f(z) = (AA^*)_{j,i}^{-1}(z) f(z).$$

This completes the proof. $\square$

Remark 3.6. This relationship provides us with the impetus for our general Fock-space Hilbert space construction. In particular, the $2N \times 2N$ positive matrix $\mathcal{S}^*\mathcal{S}$, where $\mathcal{S} = [S, \tilde{S}]$ and $S = (S_0, \ldots, S_{N-1})$, $\tilde{S} = (\tilde{S}_0, \ldots, \tilde{S}_{N-1})$ form a biorthogonal wavelet representation, has the form

$$\mathcal{S}^*\mathcal{S} = \begin{bmatrix} AA^* & I_N \\ I_N & (AA^*)^{-1} \end{bmatrix}.$$

Further, this positive matrix has *commuting* entries since the operators in the lemma are *multiplication* operators.

4. Fock space on positive matrices

There are now several Fock space constructions which appear in the literature. Typically, they allow certain identities to be represented by operators on Hilbert space by way of natural left creation operators associated with the underlying Fock space. See [5,25–27,30,32] for some different perspectives. The purpose of this section is to introduce a new Fock space construction which, we believe, may provide the appropriate framework for studying the biorthogonal wavelet representations discussed above. In any event, we find this construction to be interesting in its own right. To establish the nomenclature we use for the next two section we begin by reviewing the formulation of unrestricted Fock space. In Example 4.12 we point out how this motivating special case fits into our construction.

Example 4.1. The *full* (*unrestricted*) *Fock space* over $\mathbb{C}^N$, where N is a fixed positive integer with $N \geqslant 2$, is the orthogonal direct sum of Hilbert spaces given by

$$\mathscr{K} = \left(\sum_{k=-\infty}^{-1}{}^{\oplus} (\mathbb{C}^N)^{\otimes -k} \right) \oplus \mathbb{C} = \cdots \oplus (\mathbb{C}^N \otimes \mathbb{C}^N) \oplus (\mathbb{C}^N) \oplus \mathbb{C}.$$

The number **1** in the summand on the right (giving the copy of $\mathbb{C}$) is called the *vacuum vector* and is denoted by Ω. Let $\{\xi_1, \ldots, \xi_N\}$ be a fixed orthonormal basis for $\mathbb{C}^N$. Then $\mathscr{K}$ is an infinite-dimensional Hilbert space with orthonormal basis given by

$$\{\xi_{i_1} \otimes \cdots \otimes \xi_{i_k} \mid 1 \leqslant i_1, \ldots, i_k \leqslant N,\ k \geqslant 1\} \cup \{\Omega\}.$$

We wish to think of the infinite direct sum as extending from right to left. This is non-standard, but we believe it is helpful in understanding the action of the creation operators (see below), and it allows us to introduce notation which is less cumbersome.

The *left creation operator* L_i determined by ξ_i on $\mathscr{K}$ is defined by the actions:

$$\begin{cases} L_i(\Omega) = \xi_i, \\ L_i(\eta_k \otimes \cdots \otimes \eta_1) = \xi_i \otimes \eta_k \otimes \cdots \otimes \eta_1, \end{cases}$$

for all $k \geqslant 1$ and $\eta_1, \ldots, \eta_k \in \mathbb{C}^N$. The adjoint of L_i is the *annihilation operator* determined by ξ_i, and it acts by

$$\begin{cases} L_i^*(\Omega) = 0, \\ L_i^*(\eta_1) = \langle \eta_1 \mid \xi_i \rangle \Omega, \\ L_i^*(\eta_k \otimes \cdots \otimes \eta_1) = \langle \eta_k \mid \xi_i \rangle (\eta_{k-1} \otimes \cdots \otimes \eta_1), \end{cases}$$

for all $k \geqslant 2$ and $\eta_1, \ldots, \eta_k \in \mathbb{C}^N$. This terminology comes from theoretical physics where ‘creation’ signifies the creation of a new particle.

There is another formulation of Fock space $\mathscr{K}$ which leads to a notational simplification for us. Let $\mathbb{F}_N^+$ be the unital free semigroup on N *non-commuting* letters $\{1, 2, \ldots, N\}$ with unit e. Given w in $\mathbb{F}_N^+$, the positive integer $|w|$ is the length of the word w. The unit e corresponds to the word of length zero, or the empty word. Then one can also think of the Fock space $\mathscr{K}$ as $\ell^2(\mathbb{F}_N^+)$, where an orthonormal basis is given by the vectors $\{\xi_w : w \in \mathbb{F}_N^+\}$ corresponding to words. Thus the vectors $\xi_{i_1} \otimes \cdots \otimes \xi_{i_k}$ are identified with ξ_w where the product $w = i_1 \cdots i_k$ is in the free semigroup $\mathbb{F}_N^+$. Also, the vacuum vector is identified with ξ_e. We shall further simplify notation by referring to the vector ξ_w just by the word w. Hence the action of the creation operators is encapsulated in the short statement

$$L_i(w) = iw \quad \text{for } w \in \mathbb{F}_N^+,$$

where again we emphasize that the product iw is in the free semigroup $\mathbb{F}_N^+$. The actions of the annihilation operators are also easily described by $L_i^*(e) = 0$, and $L_i^*(jw) = w$ when $i = j$ and 0 otherwise. These operators can, in fact, be defined independent of basis (for $\xi \in \mathbb{C}^N$, an operator L_ξ can be analogously defined). We shall see this is also the case in our setting, but it is convenient to fix a basis for the analysis.

It is not hard to see that $L = (L_1, \ldots, L_N)$ forms an N-tuple of isometries with pairwise orthogonal ranges, for which the closed span of the ranges of the isometries span the orthogonal complement of the span of the vacuum vector. Equivalently, since the $L_iL_i^*$ are the range projections, this says

$$L_i^*L_j = \delta_{i,j}I \quad \text{for } 1 \leqslant i,j \leqslant N \quad \text{and} \quad \sum_{i=1}^{N} L_iL_i^* = I - P_\Omega.$$

These are the so-called *Cuntz–Toeplitz* isometries, and the C^*-algebra they generate is denoted by $\mathscr{E}_N$. The ideal generated by the rank one projection P_Ω in $\mathscr{E}_N$ determines a copy of the compact operators, and when it is factored out the Cuntz algebra $\mathcal{O}_N$ is obtained. Thus there is a tight relationship between $\mathcal{O}_N$ and the operators $L = (L_1, \ldots, L_N)$.

Note 4.2. The reader will notice that in the previous example, and for the next two sections, we have changed our notation with N-tuples of operators from $\{0, 1, \ldots, N-1\}$ to $\{1, 2, \ldots, N\}$. Unfortunately, this is the price to pay for combining the two different perspectives. In wavelet analysis the standard notation for multiresolution wavelet functions is the former (0 is for 'low frequency'), while in the realm of theoretical physics and creation operators the latter is necessary to portray the 'creation' of new particles. In any event, we hope this note will preempt any confusion.

The starting point for our general construction is an extension result for completely positive maps. First, let us recall the dichotomy between completely positive maps and positive matrices given by Choi's Lemma [13]. Let $\{e_{i,j}\}_{1 \leqslant i,j \leqslant N}$ be matrix units for the set of $N \times N$ complex matrices $\mathscr{M}_N$ corresponding to a fixed orthonormal basis $\{\xi_1, \ldots, \xi_N\}$ for $\mathbb{C}^N$. Our construction is independent of basis (see Remark 5.4), but for the sake of brevity we shall work with a fixed basis. The completely positive maps $\Phi: \mathscr{M}_N \to \mathscr{B}(\mathscr{H})$ can be identified with the positive matrices $P = [p_{i,j}] \in \mathscr{M}_N(\mathscr{B}(\mathscr{H}))$, where the correspondence is given by

$$P = \Phi^{(N)}([e_{i,j}]) = [\Phi(e_{i,j})].$$

We shall call $P = [\Phi(e_{i,j})]$ the *Choi matrix* associated with Φ. Every such completely positive map can be extended in a natural way to the matrix algebras $\mathcal{M}_{N^k}$.

Lemma 4.3. *Let Φ: $\mathcal{M}_N \to \mathcal{B}(\mathcal{H})$ be a completely positive map. Then there is a unique map $\tilde{\Phi}$: $\bigcup_{k\geqslant 1} \mathcal{M}_{N^k} \to \mathcal{B}(\mathcal{H})$ such that*

$$\tilde{\Phi}(a \otimes b) = \Phi(a)\tilde{\Phi}(b)$$

whenever $a \in \mathcal{M}_N$ and $b \in \bigcup_{k\geqslant 1} \mathcal{M}_{N^k}$. In particular, for $a_1, \dots, a_k \in \mathcal{M}_N$ we have $\tilde{\Phi}(a_1 \otimes \cdots \otimes a_k) = \Phi(a_1)\cdots\Phi(a_k)$. The natural extension of Φ to UHF_{N^∞} *is not necessarily bounded.*

Proof. The definition of $\tilde{\Phi}$ is forced upon us by the conclusion. In fact, $\tilde{\Phi}$ is determined by a sequence of completely positive maps on the algebras $\mathcal{M}_{N^k}$. The matrices $e_{i_1 i'_1} \otimes \cdots \otimes e_{i_k i'_k}$, for $1 \leqslant i_j, i'_j \leqslant N$ and $1 \leqslant j \leqslant k$, form a set of matrix units for the k-fold tensor algebra $\mathcal{M}_{N^k} \cong \mathcal{M}_N^{\otimes k}$. (We use the standard identification of matrices in $\mathcal{M}_{N^k}$ with tensors found in such texts as [36].) As above, let $P = [p_{i,j}] = \Phi^{(N)}([e_{i,j}])$. For $k \geqslant 1$, define maps Φ_k: $\mathcal{M}_{N^k} \to \mathcal{B}(\mathcal{H})$ by

$$\Phi_k(e_{i_1 i'_1} \otimes \cdots \otimes e_{i_k i'_k}) = \Phi(e_{i_1 i'_1}) \cdots \Phi(e_{i_k i'_k}) = p_{i_1 i'_1} \cdots p_{i_k i'_k}.$$

Each of these maps is completely positive by Choi's Lemma since

$$\Phi_k^{(N^k)}([e_{i_1 i'_1} \otimes \cdots \otimes e_{i_k i'_k}]) = [p_{i_1 i'_1} \cdots p_{i_k i'_k}] \cong P^{\otimes k} \geqslant 0,$$

where the indices in the first two matrices satisfy $1 \leqslant i_j, i'_j \leqslant N$ and $1 \leqslant j \leqslant k$. Thus for $a \in \mathcal{M}_{N^k}$, define $\tilde{\Phi}(a) = \Phi_k(a)$. Then $\tilde{\Phi}$: $\bigcup_{k\geqslant 1} \mathcal{M}_{N^k} \to \mathcal{B}(\mathcal{H})$ is a map which has the desired properties. Uniqueness clearly follows from these properties.

When $\bigcup_{k\geqslant 1} \mathcal{M}_{N^k}$ is regarded as an increasing union (given by unital embeddings) which generates UHF_{N^∞}, the natural extension of Φ to UHF_{N^∞} will be unbounded in general. Indeed, the identity in this algebra is obtained as a limit $I = \lim_{k\to\infty} I_{N^k}$, and for $k \geqslant 1$, $I_{N^k} \cong I_N^{\otimes k}$. Hence we would have

$$\tilde{\Phi}(I) = \lim_{k\to\infty} \tilde{\Phi}(I_{N^k}) = \lim_{k\to\infty} \Phi(I_N)^k,$$

which may be unbounded if Φ is not completely contractive. $\square$

We are not concerned with the viability of an extension to UHF_{N^∞} since it is not necessary for the Fock-space construction. The crucial point for us is that completely positive maps on $\mathcal{M}_N$ can be extended to the 'pre-UHF_{N^∞}' algebras $\mathcal{M}_{N^k}$. We will let Φ denote the map *and* its extension when there is no confusion.

Construction 4.4. Let $\mathcal{H}$ be a Hilbert space. Heuristically, our construction can be thought of as formally taking the tensor product of unrestricted Fock space with $\mathcal{H}$, then defining a 'twisted' inner product on the result by using a completely positive map from the complex matrices into $\mathscr{B}(\mathscr{H})$ (or, if you like, a positive matrix with entries in $\mathscr{B}(\mathscr{H})$).

We define the *N-variable pre-Fock space over* $\mathscr{H}$ to be the vector space of finite sums

$$\mathscr{T}_N(\mathscr{H}) = \left\{ \sum_{|w| \leqslant k} w \otimes h_w \,\middle|\, w \in \mathbb{F}_N^+,\ k \geqslant 1,\ h_w \in \mathscr{H} \right\},$$

where philosophically a vector $(i_1 \cdots i_k) \otimes h$, with $i_1, \ldots, i_k \in \mathbb{F}_N^+$, corresponds to the vector $\xi_{i_1} \otimes \cdots \otimes \xi_{i_k} \otimes h$ in $(\mathbb{C}^N)^{\otimes k} \otimes \mathscr{H}$. Let Φ: $\mathscr{M}_N \to \mathscr{B}(\mathscr{H})$ be a completely positive map. Define a form $\langle \cdot | \cdot \rangle_\Phi$: $\mathscr{T}_N(\mathscr{H}) \times \mathscr{T}_N(\mathscr{H}) \to \mathbb{C}$ in the following manner: For w, w' in $\mathbb{F}_N^+$ and h, h' in $\mathscr{H}$,

(i) $\langle e \otimes h | e \otimes h' \rangle_\Phi = \langle h \,|\, h' \rangle$;
(ii) if $|w| \neq |w'|$, then $\langle w \otimes h \,|\, w' \otimes h' \rangle_\Phi = 0$;
(iii) if $w = i_1 \cdots i_k$ and $w' = i'_1 \cdots i'_k$, then

$$\langle w \otimes h \,|\, w' \otimes h' \rangle_\Phi = \langle h \,|\, \Phi(e_{i_1 i'_1} \otimes \cdots \otimes e_{i_k i'_k}) h' \rangle.$$

Then extend $\langle \cdot | \cdot \rangle_\Phi$ to $\mathscr{T}_N(\mathscr{H}) \times \mathscr{T}_N(\mathscr{H})$ as linear in the first variable and conjugate linear in the second.

Theorem 4.5. *The form* $\langle \cdot | \cdot \rangle_\Phi$ *is positive semi-definite on* $\mathscr{T}_N(\mathscr{H})$.

Proof. Let $x \in \mathscr{T}_N(\mathscr{H})$ be a *finite* sum of the form

$$x = \sum_{k \geqslant 0} \sum_{|w|=k} w \otimes h_w.$$

As above, let $P = [p_{i,j}] = [\Phi(e_{i,j})] \in \mathscr{M}_N(\mathscr{B}(\mathscr{H}))$ be the positive Choi matrix determined by Φ. Recall from the previous lemma that the extended Φ satisfies $\Phi(e_{i_1 i'_1} \otimes \cdots \otimes e_{i_k i'_k}) = p_{i_1 i'_1} \cdots p_{i_k i'_k}$. Thus

$$\begin{aligned}
\langle x | x \rangle_\Phi &= \sum_{k,l \geqslant 0} \sum_{\substack{|w|=k \\ |w'|=l}} \langle w \otimes h_w \,|\, w' \otimes h_{w'} \rangle_\Phi \\
&= \sum_{k \geqslant 0} \sum_{|w|=k} \langle w \otimes h_w \,|\, w' \otimes h_{w'} \rangle_\Phi \\
&= \sum_{k \geqslant 0} \sum_{1 \leqslant j \leqslant k} \sum_{1 \leqslant i_j, i'_j \leqslant N} \langle h_{i_1 \cdots i_k} \,|\, (p_{i_1 i'_1} \cdots p_{i_k i'_k}) h_{i'_1 \cdots i'_k} \rangle.
\end{aligned}$$

However, if we let $z_k = (h_w)_{|w|=k} \in \mathscr{H}^{(N^k)}$, this quantity becomes

$$\langle x \mid x \rangle_\Phi = \sum_{k \geqslant 0} \langle z_k \mid P^{\otimes k} z_k \rangle_{\mathscr{H}^{(N^k)}} \geqslant 0. \qquad \square$$

Definition 4.6. Let $\mathscr{N}_\Phi = \{x \in \mathscr{T}_N(\mathscr{H}) \mid \langle x|x \rangle_\Phi = 0\}$ be the kernel of the form $\langle \cdot | \cdot \rangle_\Phi$. Since every positive semi-definite form satisfies the Cauchy–Schwarz inequality, $\mathscr{N}_\Phi$ is a subspace. We define the *Fock space of* Φ (*or* $P = [\Phi(e_{i,j})]$) *over* $\mathscr{H}$ to be the Hilbert space completion

$$\mathscr{F}_N(\mathscr{H}, \Phi) = \overline{\mathscr{T}_N(\mathscr{H}) / \mathscr{N}_\Phi}^{\langle \cdot | \cdot \rangle_\Phi}.$$

Note 4.7. The inner product on $\mathscr{F}_N(\mathscr{H}, \Phi)$ is given by

$$\langle x + \mathscr{N}_\Phi \mid y + \mathscr{N}_\Phi \rangle = \langle x \mid y \rangle_\Phi.$$

We refer to the space $e \otimes \mathscr{H} + \mathscr{N}_\Phi$ as the *vacuum space* of $\mathscr{F}_N(\mathscr{H}, \Phi)$. Recall from the definition of the inner product that orthogonality is preserved at the level of the vacuum space, unlike perhaps for words of larger length. Hence there is no ambiguity in identifying it with $\mathscr{H}$. Further, notice that in the notation $\mathscr{F}_N(\mathscr{H}, \Phi)$, reference to the space $\mathscr{H}$ is really redundant, since it is fixed when Φ is given. In other words, the construction is totally determined by the completely positive map (equivalently, by the associated positive matrix).

The kernel $\mathscr{N}_\Phi$ can be explicitly identified in terms of P, in fact the kernel of P. This is implicit in the previous proof, as is a Fourier expansion for vectors in $\mathscr{F}_N(\mathscr{H}, \Phi)$. This all follows from the existence of projections onto 'words of different lengths'. For $k \geqslant 0$, let P_k be the map defined on finite sums $x = \sum_{w \in \mathbb{F}_N^+} w \otimes h_w + \mathscr{N}_\Phi$ (in other words, finitely many h_w are non-zero) by

$$P_k x = \sum_{|w|=k} w \otimes h_w + \mathscr{N}_\Phi.$$

Lemma 4.8. *The maps* P_k, *for* $k \geqslant 0$, *extend to projections on the Fock space* $\mathscr{F}_N(\mathscr{H}, \Phi)$ *with pairwise orthogonal ranges. Further, we have* $I = \sum_{k \geqslant 0}^{\oplus} P_k$, *where the infinite sum is the limit in the strong operator topology.*

Proof. Each P_k is clearly an idempotent. Let $k \geqslant 0$ be fixed and let $x = \sum_{w \in \mathbb{F}_N^+} w \otimes h_w + \mathscr{N}_\Phi$ be a finite sum. Then by orthogonality,

$$\begin{aligned} ||P_k x||^2 &= \sum_{|w|=k=|w'|} \langle w \otimes h_w \mid w' \otimes h_{w'} \rangle \\ &\leqslant \sum_{k \geqslant 0} \sum_{|w|=k=|w'|} \langle w \otimes h_w \mid w' \otimes h_{w'} \rangle = ||x||^2. \end{aligned}$$

Hence, P_k extends to a contractive idempotent on $\mathscr{F}_N(\mathscr{H},\Phi)$, and as such, P_k is a (self-adjoint) projection on $\mathscr{F}_N(\mathscr{H},\Phi)$. Since the ranges of the P_k are pairwise orthogonal, the strong-operator-topology limit $\sum_{k\geqslant 0}^{\oplus} P_k$ exists. However, this operator acts as the identity on a dense subset of $\mathscr{F}_N(\mathscr{H},\Phi)$, so it *is* in fact the identity operator. $\square$

Corollary 4.9. *Every vector x in $\mathscr{F}_N(\mathscr{H},\Phi)$ has a representation of the form*

$$x = \sum_{k\geqslant 0} P_k x = \sum_{w\in\mathbb{F}_N^+} w\otimes h_w + \mathscr{N}_\Phi.$$

This representation is unique up to choice of the vectors

$$P_k x = \sum_{|w|=k} w\otimes h_w + \mathscr{N}_\Phi.$$

More can be said in the Cuntz–Toeplitz case: in unrestricted Fock space vectors have *bona fide* Fourier expansions, since the vectors corresponding to words form an orthonormal basis. In our setting this can be seen as a relic of $P = I_N$ having no kernel in that case (this example is discussed further below). More generally, the positive matrix P will have non-trivial kernel, thus limiting the uniqueness of the Fourier expansion up to representations of the vectors $P_k x$. We can obtain a tight upper bound on the norms of such vectors.

Proposition 4.10. *Let $h_w\in\mathscr{H}$ for each word w in $\mathbb{F}_N^+$ with $|w| = k$. Then*

$$\left|\left| \sum_{|w|=k} w\otimes h_w + \mathscr{N}_\Phi \right|\right|^2 \leqslant ||P||^k \left(\sum_{|w|=k} ||h_w||^2 \right).$$

Further, this estimate is best possible in the sense that it can always be asymptotically attained for some choice of vectors h_w.

Proof. Let $z = (h_w)_{|w|=k} \in \mathscr{H}^{(N^k)}$. Then as in the proof of Theorem 4.5 we have

$$\begin{aligned}\left|\left| \sum_{|w|=k} w\otimes h_w + \mathscr{N}_\Phi \right|\right|^2 &= \sum_{|w|=k=|w'|} \langle w\otimes h_w \mid w'\otimes h_{w'} \rangle \\ &= \sum_{j=1}^{k} \sum_{1\leqslant i_j, i'_j\leqslant N} \langle h_{i_1\cdots i_k} \mid (p_{i_1 i'_1}\cdots p_{i_k i'_k}) h_{i'_1\cdots i'_k} \rangle \\ &= \langle z \mid P^{\otimes k} z \rangle \leqslant ||P||^k ||z||^2.\end{aligned}$$

This establishes the desired inequality. It is best possible since the vectors $z \in \mathscr{H}^{(N^k)}$ can be chosen to approximate the norm of $P^{\otimes k}$ $\square$.

These projections also allow us to illustrate further the dependence of the Fock spatial structure on the matrix P in that they lead to a lucid identification of the kernel.

Theorem 4.11. *The kernel $\mathscr{N}_\Phi$ is the closed span of the pairwise orthogonal subspaces $P_k\mathscr{N}_\Phi$, for $k \geqslant 1$, given by*

$$\begin{aligned} P_k\mathscr{N}_\Phi &= \left\{ \sum_{|w|=k} w \otimes h_w \,\middle|\, (h_w)_{|w|=k} \in \ker P^{\otimes k} \right\} \\ &= \left\{ \sum_{|w|=k} w \otimes h_w \,\middle|\, (h_w)_{|w|=k} \simeq x_1 \otimes \cdots \otimes x_k \text{ with } \right. \\ &\qquad \left. x_1, \ldots, x_k \in \mathscr{H}^{(N)} \text{ and some } x_i \in \ker P \right\}. \end{aligned}$$

In particular, $\mathscr{N}_\Phi$ is completely determined by the kernel of P.

Proof. With the existence of the projections P_k proved in the previous lemma, we have $\mathscr{N}_\Phi = \sum_{k\geqslant 0}^{\oplus} P_k\mathscr{N}_\Phi$. The characterization of the subspaces $P_k\mathscr{N}_\Phi$ in terms of P comes as a direct consequence of the proof of Theorem 4.5. Indeed, from the end of that proof we see that $P_k\mathscr{N}_\Phi$ is determined by $\ker P^{\otimes k}$, and this subspace has the desired form. $\square$

We finish this section by observing the ways in which unrestricted Fock space is captured by our construction.

Example 4.12. Let $P = I_N$ be the identity matrix in $\mathscr{M}_N(\mathbb{C})$. Then P is the Choi matrix for the completely positive map Φ: $\mathscr{M}_N \to \mathbb{C}$ defined by $\Phi(e_{i,j}) = 1$ if $i = j$ and 0 if $i \neq j$. The corresponding Fock space $\mathscr{F}_N(\mathbb{C}, I_N)$ is the standard unrestricted version. Indeed, since the kernel of P is trivial, the null set $\mathscr{N}_\Phi = \{0\}$. Further, for words $w = i_1 \cdots i_k \in \mathbb{F}_N^+$ and letters $1 \leqslant i, j \leqslant N$ we have

$$\langle iw \otimes \Omega \,|\, jw \otimes \Omega \rangle_\Phi = \langle \Omega \,|\, (p_{i,j} p_{i_1,i_1} \cdots p_{i_k,i_k})\Omega \rangle.$$

It follows that the vectors $\{w \otimes \Omega\}_{w \in \mathbb{F}_N^+}$ form an orthonormal basis for the space, and structurally $\mathscr{K} = \ell^2(\mathbb{F}_N^+)$ is obtained simply by identifying basis vectors $w \otimes \Omega$ with w for $w \in \mathbb{F}_N^+$ (see Example 4.1).

The construction also yields $\mathscr{K} = \ell^2(\mathbb{F}_N^+)$ through what can be considered as the biorthogonal setting. We will say more about this at the end of the next section. Let

E be the matrix in $\mathcal{M}_2(\mathbb{C})$ with a one in each entry and let $P = I_N \otimes E$. Then the construction of $\mathcal{F}_{2N}(\mathbb{C}, P)$ yields $\mathcal{K} = \ell^2(\mathbb{F}_N^+)$ once again. In this case, a non-trivial kernel for the matrix P leads to a non-trivial null set $\mathcal{N}_\Phi$. In particular, for words w in $\mathbb{F}_{2N}^+$ and letters $i \in \{1, \ldots, 2N\}$ with $1 \leqslant i \leqslant N$, the vectors

$$iw \otimes \Omega - (i + N)w \otimes \Omega$$

belong to $\mathcal{N}_\Phi$, since $p_{i,i} = p_{i+N,i+N} = p_{i+N,i} = p_{i,i+N} = 1$. This collapse, together with the other entries of P, shows that the vectors $\{w \otimes \Omega + \mathcal{N}_\Phi\}_{w \in \mathbb{F}_N^+}$ form an orthonormal basis for $\mathcal{F}_{2N}(\mathbb{C}, P)$. In other words, $\mathcal{K} = \ell^2(\mathbb{F}_N^+)$ is obtained structurally, with an obvious identification of orthonormal bases. A similar analysis also shows that the construction for $\mathcal{F}_{2N}(\mathcal{H}, I_{kN} \otimes E)$, where $\mathcal{H}$ is k-dimensional Hilbert space, yields $\ell^2(\mathbb{F}_N^+)^{(k)}$.

5. Creation operators

The Fock spaces from the previous section yield creation operators which reduce to the Cuntz–Toeplitz isometries in the unrestricted Fock space setting.

Definition 5.1. The *left creation operators* $T = (T_1, \ldots, T_N)$ on $\mathcal{F}_N(\mathcal{H}, \Phi)$ are linear transformations defined by

$$T_i(w \otimes h + \mathcal{N}_\Phi) = (iw) \otimes h + \mathcal{N}_\Phi,$$

where once again the product iw is the free semigroup $\mathbb{F}_N^+$.

Since there is non-trivial null space in general, we must check that these operators are well-defined.

Proposition 5.2. *The operators $T = (T_1, \ldots, T_N)$ are well defined since $T_i \mathcal{N}_\Phi \subseteq \mathcal{N}_\Phi$ for $1 \leqslant i \leqslant N$.*

Proof. It suffices to check that $T_i P_k \mathcal{N}_\Phi \subseteq P_{k+1} \mathcal{N}_\Phi$. Let $x = P_k x = \sum_{|w|=k} w \otimes h_w + \mathcal{N}_\Phi$ belong to $\mathcal{N}_\Phi$, and put $z = (h_w)_{|w|=k} \in \mathcal{H}^{(N^k)}$. Then $z \in \ker P^{\otimes k}$ by Theorem 4.11, and we have

$$\langle T_i x \mid T_i x \rangle = \sum_{|w|=k=|w'|} \langle iw \otimes h_w \mid iw' \otimes h_{w'} \rangle_\Phi = \langle z \mid p_{i,i}^{(N^k)} P^{\otimes k} z \rangle = 0.$$

Thus $T_i x$ belongs to $\mathcal{N}_\Phi$, and it follows that T_i is well defined. $\square$

For ease of presentation we shall suppress reference to the kernel $\mathcal{N}_\Phi$ for the rest of this section.

Remark 5.3. Our notion of the creation operators is in principle similar to, but yet quite different from, others in the literature. One instance of this notion is the one used in [33,34] in the construction of covariant representations of tensor algebras, such as the Cuntz–Pimsner algebra. The basic concepts in [33,34] are an inner product $\langle \cdot | \cdot \rangle$ on a module E taking values in a C^*-algebra $\mathfrak{A}$, and a completely positive mapping Φ: $\mathfrak{A} \to \mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is a Hilbert space. Then an extended inner product is defined on a tensor algebra over E, starting with $E \otimes \mathcal{H}$, as follows:

$$\langle a \otimes \xi \,|\, b \otimes \eta \rangle_{\text{new}} \coloneqq \langle \xi \,|\, \Phi(\langle a \,|\, b \rangle)\eta \rangle, \quad \text{for } a, b \in E \text{ and } \xi, \eta \in \mathcal{H}.$$

However, our construction and setting are somewhat different. While we do start with a completely positive map Φ: $\mathcal{M}_N \to \mathcal{B}(\mathcal{H})$, our construction yields a *bona fide* inner product which is defined by a certain natural extension of the map $\tilde{\Phi}$ to the 'pre-UHF$_{N^\infty}$' algebras $\mathcal{M}_{N^k}$ (see Lemma 4.3). Further, in our most general case the creation operators we get are not necessarily bounded. Moreover, we need the details from the intermediate steps of the construction.

We also remark on the basis independence of this construction.

Remark 5.4. The reader will notice that, for a fixed completely positive map Φ, the notation $\mathcal{F}_N(\mathcal{H}, \Phi)$ makes no reference to the basis for $\mathbb{C}^N$ used in our construction. The reason for this is that the creation operators obtained by using different orthonormal bases are unitarily equivalent. To see this, let $\{\xi_1, \dots, \xi_N\}$ and $\{\eta_1, \dots, \eta_N\}$ be orthonormal bases for $\mathbb{C}^N$. Then the two Fock spaces constructed will be spanned by vectors of the form, respectively, $\xi_{i_1} \otimes \cdots \otimes \xi_{i_k} \otimes h$ and $\eta_{i_1} \otimes \cdots \otimes \eta_{i_k} \otimes h$ (again, suppressing reference to the kernel). Let U be the unitary between these two spaces which identifies such spanning vectors (it is a unitary since the inner product is computed in the same way). For $1 \leqslant i \leqslant N$, let T_i and S_i be the ith creation operators on the first and second space. Then for vectors $x = \xi_{i_1} \otimes \cdots \otimes \xi_{i_k} \otimes h$ we have

$$\begin{aligned} U^* S_i U x &= U^* S_i(\eta_{i_1} \otimes \cdots \otimes \eta_{i_k} \otimes h) \\ &= U^*(\eta_i \otimes \eta_{i_1} \otimes \cdots \otimes \eta_{i_k} \otimes h) = T_i x. \end{aligned}$$

Thus, in this sense there is no ambiguity in using the notation $\mathcal{F}_N(\mathcal{H}, \Phi)$, given the completely positive map Φ.

Through the behavior of the inner product, we can describe the action of T_i^* on the spanning vectors. Since

$$\langle T_i^*(e \otimes h) \,|\, w \otimes h' \rangle = \langle e \otimes h \,|\, (iw) \otimes h' \rangle = 0,$$

for all words $w \in \mathbb{F}_N^+$, we have $T_i^* \mathscr{H} = 0$. Further $\langle T_i^*(w \otimes h) \mid w' \otimes h' \rangle = 0$ unless $|w| = |w'| + 1$. In this case, if $w = i_1 \cdots i_k$ and $w' = i_2' \cdots i_k'$, we have

$$\begin{aligned} \langle T_i^*(w \otimes h) \mid w' \otimes h' \rangle &= \langle w \otimes h \mid (iw') \otimes h' \rangle \\ &= \langle h \mid (p_{i_1 i} p_{i_2 i_2'} \cdots p_{i_k i_k'}) h' \rangle. \end{aligned}$$

Let us summarize these actions in terms of our Fourier projections.

Proposition 5.5. *For $1 \leqslant i \leqslant N$, we have $P_0 T_i = 0$ and $T_i P_k = P_{k+1} T_i$ for $k \geqslant 0$.*

Proof. This simply follows from the above analysis. Indeed, it was observed that T_i^* annihilates $\operatorname{Ran} P_0 = \mathscr{H}$. Further, since $I = \sum_{j \geqslant 0}^{\oplus} P_j$ we have

$$T_i P_k = P_{k+1} T_i P_k = P_{k+1} T_i \sum_{j \geqslant 0}^{\oplus} P_j = P_{k+1} T_i. \qquad \square$$

When the T_i are isometries with pairwise orthogonal ranges, the act of 'pushing out' then 'pulling back' is given by $T_i^* T_j = \delta_{i,j} I$. In general this action will not be so clean, since the inner product twists each time T_i or T_i^* is applied. Nonetheless, the action can be readily described in terms of the matrix P.

Theorem 5.6. *Let $w \in \mathbb{F}_N^+$ and $h \in \mathscr{H}$. Then for $1 \leqslant i,j \leqslant N$ we have*

$$T_i^* T_j (w \otimes h) = w \otimes p_{i,j} h.$$

Proof. Since the vectors $w \otimes h$ span (not necessarily orthogonally) the Fock space $\mathscr{F}_N(\mathscr{H}, \Phi)$, it suffices to examine inner products $\langle T_i^* T_j (w \otimes h) \mid w' \otimes h' \rangle$ for $w, w' \in \mathbb{F}_N^+$ and $h, h' \in \mathscr{H}$. If w and w' are words of different lengths, this inner product is clearly zero. On the other hand, if $w = i_1 \cdots i_k$ and $w' = i_1' \cdots i_k'$ we have

$$\begin{aligned} \langle T_i^* T_j (w \otimes h) \mid w' \otimes h' \rangle &= \langle (jw) \otimes h \mid (iw') \otimes h' \rangle \\ &= \langle h \mid (p_{ji} p_{i_1 i_1'} \cdots p_{i_k i_k'}) h' \rangle \\ &= \langle p_{i,j} h \mid (p_{i_1 i_1'} \cdots p_{i_k i_k'}) h' \rangle \\ &= \langle w \otimes p_{i,j} h \mid w' \otimes h' \rangle. \end{aligned}$$

The result now follows from the existence of the Fourier expansions for vectors in $\mathscr{F}_N(\mathscr{H}, \Phi)$. $\square$

Thus, at least formally, the transformations $T_i^* T_j$ can be thought of as the tensor product of the identity on unrestricted Fock space together with the operator $p_{i,j}$. We can use this computation to discuss boundedness of the creation operators.

Corollary 5.7. *When the representation of* $\mathbb{F}_N^+$ *on* $\mathcal{F}_N(\mathcal{H}, \Phi)$ *determined by the creation operators* $T = (T_1, \ldots, T_N)$ *yields bounded operators, the completely bounded norm of the map* Φ *is estimated by*

$$||\Phi||_{\mathrm{cb}} \leqslant \sum_{i=1}^{n} ||T_i||^2.$$

Proof. Since Φ is completely positive and $\Phi(e_{i,i}) = p_{i,i}$, we have the inequality

$$||\Phi||_{\mathrm{cb}} = ||\Phi(I_N)|| = \left|\left| \sum_{i=1}^{N} \Phi(e_{i,i}) \right|\right| \leqslant \sum_{i=1}^{N} ||p_{i,i}||.$$

However, from the theorem it follows that $||p_{i,i}|| = ||T_i^* T_i|_{\mathcal{H}}|| \leqslant ||T_i||^2 < \infty$. Whence, the desired estimate is obtained. $\square$

There is a partial converse of this result which contains all the cases we are interested in.

Corollary 5.8. *Let* $T = (T_1, \ldots, T_N)$ *be the creation operators on* $\mathcal{F}_N(\mathcal{H}, \Phi)$, *where* Φ *has commutative range. Then for* $1 \leqslant i \leqslant N$, *the norm of* T_i *is given by* $||T_i|| = ||p_{i,i}||^{\frac{1}{2}}$.

Proof. By Proposition 5.5, we have $T_i^* T_i P_k = T_i^* P_{k+1} T_i = P_k T_i^* T_i$ for $k \geqslant 0$. Hence $T_i^* T_i$ is diagonal with respect to the decomposition $\mathcal{F}_N(\mathcal{H}, \Phi) = \sum_{k \geqslant 0}^{\oplus} P_k \mathcal{F}_N(\mathcal{H}, \Phi)$, and as such we obtain the norm identity

$$||T_i||^2 = ||T_i^* T_i|| = \sup_{k \geqslant 0} ||T_i^* T_i P_k||.$$

Let $k \geqslant 0$ and let h_w be vectors in $\mathcal{H}$ for each $|w| = k$. If $x = \sum_{|w|=k} w \otimes h_w$, then from the theorem we have

$$\begin{aligned} ||T_i^* T_i x||^2 &= \sum_{|w|=k=|w'|} \langle w \otimes p_{i,i} h_w \mid w' \otimes p_{i,i} h_{w'} \rangle \\ &= \left\langle p_{i,i}^{(N^k)} z \,\middle|\, P^{\otimes k} p_{i,i}^{(N^k)} z \right\rangle, \end{aligned}$$

where $z = (h_w)_{|w|=k}$ belongs to $\mathcal{H}^{(N^k)}$ and $p_{i,i}^{(N^k)}$ is the diagonal matrix in $\mathcal{M}_{N^k}(\mathcal{B}(\mathcal{H}))$ with $p_{i,i}$ down the diagonal. However, since the $p_{i,j}$ commute we arrive at the inequality

$$p_{i,i}^{(N^k)} P^{\otimes k} p_{i,i}^{(N^k)} \leqslant ||p_{i,i}||^2 P^{\otimes k}.$$

This follows from the fact that if R and S are commuting positive operators, then

$$0 \leqslant RSR = \sqrt{S} R^2 \sqrt{S} \leqslant ||R||^2 S. \tag{14}$$

Thus we have

$$||T_i^* T_i x||^2 \leqslant ||p_{i,i}||^2 \langle z \mid P^{\otimes k} z \rangle = ||p_{i,i}||^2 ||x||^2,$$

so that $||T_i^* T_i P_k|| \leqslant ||p_{i,i}||$. Since $T_i^* T_i P_0 = p_{i,i}$, it follows that $||T_i^* T_i|| = ||p_{i,i}||$, as required. $\square$

Remark 5.9. The commutativity assumption in Corollary 5.8 is essential. The reader can check that, for the case when S in (14) is a projection, then the operator inequality (14) holds if *and only if* R and S are commuting.

We finish by pointing out properties of the creation operators which are naturally determined by biorthogonal wavelet representations.

Corollary 5.10. *Let* $S = (S_0, \dots, S_{N-1})$ *and* $\tilde{S} = (\tilde{S}_0, \dots, \tilde{S}_{N-1})$ *form a biorthogonal wavelet representation on* $\mathscr{H} = L^2(\mathbb{T})$ *with invertible loop matrices* A *and* $\tilde{A}$. *Let* $\mathscr{S} = [S\ \tilde{S}]$ *be a row matrix. Let* $P = \mathscr{S}^* \mathscr{S}$ *be the matrix in* $\mathscr{M}_{2N}(\mathscr{B}(\mathscr{H}))$ *determined by the representation, as in Remark* 3.6. *Let*

$$T = (T_1, \dots, T_N, \tilde{T}_1, \dots, \tilde{T}_N)$$

be the creation operators acting on $\mathscr{F}_{2N}(\mathscr{H}, P)$. *Then for* $1 \leqslant i,j \leqslant N$, *we have*

(i) $T_i^* T_j|_{\mathscr{H}} = S_{i-1}^* S_{j-1} = (AA^*)_{i,j}$,

(ii) $\tilde{T}_i^* \tilde{T}_j|_{\mathscr{H}} = \tilde{S}_{i-1}^* \tilde{S}_{j-1} = (AA^*)_{i,j}^{-1}$,

(iii)
$$\tilde{T}_i^* T_j|_{\mathscr{H}} = T_i^* \tilde{T}_j|_{\mathscr{H}} = \begin{cases} I, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Remark 5.11. The corollary yields an esthetically pleasing relationship between the biorthogonal wavelet representations and the creation operators they determine. Indeed, the work of the last two section shows that the $S, \tilde{S}$ system completely determines the Fock-space structure, and the actions of the creation operators. However, we are still trying to get a handle on what this all means for these representations. It was observed in Example 4.12 essentially how to obtain the Cuntz–Toeplitz isometries as they sit inside the biorthogonal class. But this is a little bit misleading, for, if the previous corollary is applied with the matrix P defined by a representation of $\mathcal{O}_N$, then the Cuntz–Toeplitz isometries with *infinite multiplicity* are obtained. Perhaps our construction, when applied to the biorthogonal wavelet representations, somehow yields the appropriate creation operators for the representations repeated with infinite multiplicity? Or maybe if one wishes to study creation operators associated with these representations, the infinite multiplicity setting is a necessity? In any event, there are a number of open problems related to this class of creation operators.

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